



## ÜNİVERSİTEPARK Bülten | Bulletin

ISSN: 2147-351X | e-ISSN: 2564-8039 | [www.unibulletin.com](http://www.unibulletin.com)

### ÜNİVERSİTEPARK Bülten | Bulletin • Volume 9 • Issue 1 • 2020

#### Identification of Student Thinking Error Patterns in Construction of Mathematical Proof

Anton Prayitno and Febi Dwi Widayanti

**To cite this article:** Prayitno, A., Widayanti, F. D. (2020). Identification of Student Thinking Error Patterns in Construction of Mathematical Proof. *Üniversitepark Bülten*, 9(1), 7-14.

**To link to this article:** <http://dx.doi.org/10.22521/unibulletin.2020.91.1>

**Anton Prayitno**, Wisnuwardhana University, Indonesia. (email: [arsedi2003@gmail.com](mailto:arsedi2003@gmail.com))

**Febi Dwi Widayanti**, Wisnuwardhana University, Indonesia.

## Identification of Student Thinking Error Patterns in Construction of Mathematical Proof

ANTON PRAYITNO and FEBI DWI WIDAYANTI

### Abstract

The purpose of this study is to identify patterns of student thinking errors in constructing mathematics proof. Therefore, the study employed a mixed-method approach, combining both quantitative and qualitative paradigms. The subjects of this study were second-generation mathematics education students at Wisnuwardhana University and Kanjuruhan University, Indonesia. The results obtained showed that the pattern of students' thinking errors in the construction of mathematical proof was due to insufficient initial knowledge about the definition of proof, and difficulties in connecting the concepts of rational numbers when operating two rational numbers.

**Keywords:** Student thinking error, error pattern, mathematical proof.



DOI: 10.22521/unibulletin.2020.91.1

---

UNIBULLETIN • ISSN 2147-351X • e-ISSN 2564-8039

Copyright © 2020 by ÜNİVERSİTEPARK

unibulletin.com

## Introduction

The importance of mathematics proof has been the subject of review by a number of scholarly experts such as Dickerson (2008), who stated the essential role of mathematics proof because proof is the basis of and an inseparable part of mathematics. Also, Blanton and Stylianou (2003) and Samkof, Lai, and Weber (2012) stated that students need to improve their knowledge in the construction of mathematics proof. Additionally, Cirillo (2009) and Köğçe, Aydın, and Yıldız (2010) stated the assumption that proof is considered an essential component for working, communicating, and understanding mathematics in depth. Therefore, students must possess the ability to construct and understand mathematical proofs as it is not otherwise possible to learn mathematics to any significant degree without such proficiency. Students' ability to complete proof is related to thinking, and mathematics proof is at the core of mathematical thinking (Cheng & Lin, 2009).

Mathematics proof is formal and deductive, and a formal written proof represents the final stage of mathematical thinking. Tall (2008a, 2008b) developed a three-way theory of mathematical thinking used to describe student thinking in the construction of mathematical proof, namely (1) "embodied," the existence of real interactions that are developed based on experience, (2) "symbolic," symbolic manipulation that functions well as formal processes and concepts, and (3) "builds a system," which is based on formal evidence in order to construct a coherent theory.

The basic educational course that can be used to construct proof in mathematics education is calculus. In Indonesia, almost all students enrolled in the mathematics education department learn calculus during the first semester of their university education. If students can learn to master how to understand mathematical proof, they can go on to construct proof to an advanced level of mathematics. However, if students experience difficulties and make errors in constructing a first-level mathematical proof, then it will negatively impact on the next proof material, for example, proof in real analysis and algebraic proof. Therefore, if the initial error is not resolved, it will impact on the next mathematical proof. The learning of proof is a simple concept that can lead students to go on to develop more complex mathematical proofs (Weber & Alcock, 2004). Researchers have examined the difficulties experienced by students and their thinking processes in the construction of mathematical proofs. The results of such research (e.g., Alcock, 2010; Batanero, Godino, & Roa, 2004) have shown that the difficulties students experience when writing proof are due to deficiencies in their understanding of mathematical concepts. Furthermore, the failure of students to construct mathematical proof lies in not being able to correctly apply their knowledge as a strategy for resolving mathematical proofs (Samkof et al., 2012).

Student difficulties experienced in the construction of mathematical proof often manifest in the form of errors (Prayitno, Rossa et al., 2018; Prayitno, Setyowati et al., 2018), with student errors in mathematical proofing seen as a reflection of their thinking. If student errors in mathematical proofs are not prevented, then it will undoubtedly impact on the student's thinking when faced with more advanced mathematical proofs. Therefore, the current study aims to identify error patterns in students' thinking in the construction of mathematical proofs.

## Methodology

The approach applied in the current study was that of a mixed-methods research that combined both quantitative and qualitative approaches. The quantitative approach processed data related to the degree to which students made mistakes in mathematical proofs, while the qualitative approach was conducted in order to trace how students' errors occurred in the construction of mathematical proofs.

Data was collected by assigning proof problems to students, and then requiring them to solve them by thinking aloud (Prayitno, 2016). The research was conducted in three stages, with data collection as the first stage in which the researcher recorded the students' behaviors when working on proofing problems in order to learn how they constructed mathematical proofs. Where an error was made in the proofing exercise, then the type of error was grouped accordingly in order to build evidence for the identification of patterns that formed from the students' errors.

The population of this study was formed by second generation mathematics education students at two higher education institutions located in Malang, Indonesia, namely Wisnuwardhana University and Kanjuruhan University. The collected data was in the form of student work, interviews conducted with the students, and field notes taken by the researcher. The collected data were then analyzed by reading the data in full, and then coding and interpreting the data in order to draw conclusions (Creswell, 2009).

## Results and Discussion

In the proof of real numbers, two constructions were applied. In the first, the students were tasked with proving rational numbers, while the second involved the proofing of odd and even numbers. On the questions about the proofing of rational numbers, the majority (66%) of students experienced some degree of difficulty and produced errors. The proof question tasked to the students was as follows:

Suppose a rational number  $\frac{a}{b}$ , with  $a$  and  $b$  as integers and  $b \neq 0$ .  
Prove that the sum of the two rational numbers is rational

In resolving such a proof, many of the students responded based on rational numbers using the analogy of numbers. In this case, the students interpretation of the rational number with  $\frac{5}{6}$  or  $\frac{1}{3}$ . When there were statements about two rational numbers, the students directly operated the two rational numbers. In this case, some of the students wrote  $\frac{5}{6} + \frac{5}{6} = \frac{10}{6}$  (see Figure 1). Likewise, other students assumed that proving the number of two rational numbers is rational can be proven by  $\frac{1}{3} + \frac{2}{5} = \frac{11}{15}$ .

bil rasional =  $\frac{a}{b}$ ,  $a \neq b =$  bil bul  $\neq b \neq 0$   
jumlah 2 bilangan adalah rasional  
 $\frac{a}{b} + \frac{a}{b} = \frac{2a}{b}$   
misal  $a = 5$   
 $b = 6$  jadi  $\frac{5}{6} + \frac{5}{6} = \frac{10}{6}$  (termasuk)  
bil rasional

**Figure 1.** Student Work (S1); Proving the Sum of Two Rational Numbers

The students began their proof by thinking about what is known to them, namely the rational number  $\frac{2}{b}$ , with  $b \neq 0$  and with  $a$  and  $b$  as integers, and then adding them together as  $\frac{a}{b} + \frac{a}{b} = \frac{2a}{b}$ . In this step, the students were unable to provide the right reasoning that  $\frac{2a}{b}$  is a rational number by providing answers that were considered absolute. The following excerpt demonstrates this case:

- P : Was  $\frac{2a}{b}$  a rational number?  
S1 : No, because the real number is  $\frac{a}{b}$ , while 2 is a natural number.  
Therefore,  $\frac{2a}{b}$  is not a rational number.

The student assumed that  $\frac{2a}{b}$  was not a rational number, because numerically  $\frac{2a}{b}$  consists of a natural number (i.e., 2) and a rational number ( $\frac{a}{b}$ ). This demonstrated the student's error in thinking about rational numbers, having misunderstood the definition of rational numbers ( $\frac{a}{b}$ ;  $a$  and  $b$  integers). If they explored in depth that  $2a$  is an integer because 2 and  $a$  are integers, they would deduce that  $\frac{2a}{b}$  is of course a rational number. Another inability to understand the definition of rational numbers was indicated by specifying  $a = 5$  and  $b = 6$ .

In the next verification step, the students' thought that  $\frac{a}{b}$  could be replaced with  $\frac{5}{6}$ , thinking that the rational numbers were defined as integers 5 and 6. In the next process, they continued by writing  $\frac{5}{6} + \frac{5}{6} = \frac{10}{6}$ , because 10 and 6 are integers and 6 is not the same as zero, therefore,  $\frac{10}{6}$  was included as a rational number. Thus, the students concluded that a number formed from two rational numbers was a rational number. In this case, the students attempted a proof of rational numbers by presenting examples of a number, assuming that the examples could resolve the proof. From their way of thinking, the students inability to understand the definition of rational numbers was clearly seen. The students' errors in constructing mathematical proof also occurred in the form of other examples (i.e.,  $\frac{5}{6}$ ). Thus, the case was used in compiling the mathematical proof. As a verification process, the verification is only valid for  $\frac{5}{6}$ , but some students did not understand that this proof applies in general. An example of the errors that the students made in mathematical proof construction are shown in Figure 2.

Pembuktian

$$\begin{array}{l} a = 1 \quad c = 2 \\ b = 3 \quad d = 5 \end{array}$$

$$\frac{1}{3} + \frac{2}{5} = \frac{5+6}{15}$$

$$= \frac{11}{15}$$

$\Rightarrow$  Rasional

**Figure 2.** Student Work (S2); Proving the Sum of Two Rational Numbers

In the process of proof, two numbers were rational, and the students thought of the two rational numbers, namely  $\frac{a}{b}$  and  $\frac{c}{d}$ . In the next process, the students assumed  $a = 1, b = 3, c = 2$  and  $d = 5$ . In this example, the students assumed that both  $\frac{1}{3}$  and  $\frac{2}{5}$  were rational numbers. Although the answer was correct, the students did not understand the general mathematical process of proving rational numbers. In this step, the students did not fully understand the mathematical proof, as the following excerpt indicates.

- P : *If  $a = -5, b = -1, c = 0,$  and  $d = -2,$  is it still valid to proof that two rational numbers formed other rational numbers?*
- S2 : *Hmm [the student seemed to think for a long time], rational numbers form  $\frac{a}{b}$ , if  $a = -5$  and  $b = 1$  means  $a / b = -5$  while  $c = 0$  and  $d = -2,$  which means that  $\frac{c}{d} = 0$  becomes  $-5 + 0 = -5,$  so it is not rational.*
- P : *How are you sure it is not rational?*
- S2 : *Because the end result is not  $\frac{a}{b}$ .*

The excerpt from this interview shows that student S2 only understood the rational number  $\frac{a}{b}$ . In other words, the student understood the rational numbers as  $\frac{2}{3}$  or  $\frac{1}{5}$ , but was unable to form rational numbers (as in  $-5$  or  $0$  are rational numbers) because  $-5$  is  $\frac{-5}{1}$ , while  $0 = \frac{0}{1}$  or  $\frac{0}{2}$ .

In the process of completion, the student only understood that  $\frac{1}{3}$  and  $\frac{2}{5}$  were rational numbers. Consequently, the student added the two numbers resulting in  $5 + \frac{6}{15} = \frac{11}{15}$ . Because the student produced  $\frac{11}{15}$ , they concluded  $\frac{11}{15}$  to be a rational number and at the same time answer the proof. Based on this way of thinking, it appears that the student then experienced an inability to understand the definition of rational numbers. At the same time, the student did not understand the definition of mathematical proof. In the student's understanding of the definition of rational numbers, they were unable to numerically interpret the form of  $a / b$ . The student only understood that  $\frac{1}{2}$  or  $\frac{5}{10}$  were rational numbers, while  $2$  or  $0$  whose divisors were  $1$  were not rational numbers.

Overall, the students were found to be weak in their proofing abilities. When faced with a problem of proof, the students would directly equate to a number. Errors in compiling the proof also

occurred for other students. In collecting evidence, they assumed that  $\frac{a}{b}$  was an irrational number, as shown in Figure 3.

1.  $\frac{a}{b}$  b bilangan bulat  $b \neq 0$   
akan dibuktikan  $\frac{a}{b}$  rasional  
Asumsi:  $b \neq 0$   
Bukti:  
diperoleh  $\frac{a}{b}$  irrasional maka  $\exists \frac{a}{b} \cdot a = b = \frac{a}{b} \cdot a$   
Sehingga  
 $\frac{a}{b} = \frac{a}{b} + b$   
 $= \frac{a}{b} \cdot \left(\frac{a}{b} a\right)$   
 $= \frac{a}{b} + b$

**Figure 3.** Student Work (S3); Proving the Sum of Two Rational Numbers

To prove that two rational numbers are rational, the student applied indirect proof; assuming that  $\frac{a}{b}$  was an irrational number, and that  $\frac{a}{b} \times a = b$ . In this case, the student experienced errors in the completion process ( $\frac{a}{b} \times a = b$ ). However, even if we multiply  $\frac{a}{b}$  with a result  $a^2b$ , such an error in operation results in successive errors in the next step. Although the thinking of the student in the next step was quite logical, the results obtained were still incorrect. The student thought that  $\frac{a}{b} + b = \frac{a}{b}$ . In this result also, the student seemed unable to operate rational numbers. Even if we are consistent with the sum operation of rational numbers, it should be  $\frac{a}{b} + b = \frac{ab^2}{b}$ . The results of  $\frac{a}{b} + b = \frac{a}{b}$  show that the students thought they should compare to the left side. When clarified by the researchers, student S3 answered as follows:

- P : What is  $\frac{a}{b} + b$ ?
- S3 :  $\frac{a}{b} + b = \frac{ab^2}{b}$
- P : How do you get  $\frac{a}{b} + b = \frac{a}{b}$ ?
- S3 : Because I changed  $b = 0$ . So it's true that  $\frac{a}{b} = \frac{a}{b}$ .

From these answers, it can be seen that the student claimed that  $b = 0$ . If we consider their assumption that  $b \neq 0$ , this is a contradiction in the assumptions made in the student's completion process. As a result, the process of their conclusion was subsequently incorrect.

## Conclusion

From the results seen of the students' work, it could be seen that the students experienced an inability to understand the definition of rational numbers. Student S1 had an error in constructing mathematical proof in the form of an example (i.e.,  $5/6$ ). Therefore, the case containing the error was subsequently used by the student in compiling their mathematical proof. In understanding student S2 on the definition of rational numbers, the student was unable to numerically interpret  $a/b$ . They only understood that  $1/2$  and  $5/10$  were rational numbers, whilst  $2$  or  $0$  whose divisors are  $1$  were not rational numbers. Likewise, student S3 made an error in operating, causing an error that occurred in the

following step. Thus, the pattern of errors in mathematics proof can be identified such as insufficient initial knowledge about the definition of proof, and the difficulty of connecting the concepts of rational numbers when operating two rational numbers.

### Acknowledgments

This research was supported and funded by DRPM, Kemeristekdikti. We thank our colleagues from Wisnuwardhana University and Kanjuruhan University who provided insight and expertise that greatly assisted the research.

### Notes

Corresponding author: ANTON PRAYITNO

### References

- Alcock, L. (2010). Interactions Between Teaching and Research: Developing Pedagogical Content Knowledge for Real Analysis. In R. Leikin & R. Zazkis (Eds.), *Learning Through Teaching Mathematics. Mathematics Teacher Education* (Vol 5., pp. 227-245). Dordrecht, Germany: Springer.
- Batanero, C., Godino, J. D., & Roa, R. (2004). Training Teachers to Teach Probability. *Journal of Statistics Education*, 12(1).
- Blanton, M. L., & Stylianou, D. A. (2003). The Nature of scaffolding in Undergraduate Transition to Formal Proof. In N. A. Pateman, B. J. Dougherty, & J. T. Zilliox (Eds.), *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education* (pp. 113-120). Honolulu, HI: University of Hawai.
- Cheng, Y.-H., & Lin, F.-L. (2009). Developing Learning Strategies for Enhancing Below Average Students' Ability in Constructing Multi-Step Geometry Proof. In F.-L. Lin, F.-J. Hsieh, G. Hanna, & M. deVilliers (Eds.), *Proceeding of The ICMI Study 19 Conference: Proof and Proving in Mathematics Education, Volume 1* (pp. 124-129). Taipei, Taiwan: National Taiwan Normal University.
- Cirillo, M. (2009). Challenges to Teaching Authentic Mathematical Proof in School Mathematics. In F.-L. Lin (Ed.), *Proceeding of The ICMI Study 19 Conference: Proof and Proving in Mathematics Education, Volume 1* (pp. 130-135). Taipei, Taiwan: National Taiwan Normal University.
- Creswell, J. W. (2009). *Research Design Qualitative, Quantitative, and Mixed Approaches* (3rd ed.). United Kingdom: Sage.
- Dickerson, D. S. (2008). *High School Mathematics Teachers' Understandings of the Purposes of Mathematical Proof* (Doctoral dissertation). Syracuse, NY: Syracuse University.
- Köğçe, D., Aydın, M., & Yıldız, C. (2010). The views of high school students about proof and their levels of proof (The case of Trabzon). *Procedia - Social and Behavioral Sciences*, 2(2), 2544-2549.
- Prayitno, A. (2016). The Characteristics of Students' Refractive Thinking about Data. In M. S. Warsono (Ed.), *Proceedings of 3rd International Conference on Research, Implementation and Education of Mathematics and Science (ICRIEMS)* (pp. 29-38). Yogyakarta, Indonesia: Universitas Negeri Yogyakarta.
- Prayitno, A., Rossa, A., Widayanti, F. D., Rahayuningsih, S., Hamid, A., & Baidawi, M. (2018). Characteristics of Students' Proportional Reasoning in Solving Missing Value Problem.

- Journal of Physics: Conference Series*, 1114(1). <https://doi.org/10.1088/1742-6596/1114/1/012021>
- Prayitno, A., Setyowati, V. L., Damayanti, N. W., Khasanah, F., Mayangsari, S. N., Mahardika, L. T.,...Pertiwi, R. I. (2018). Performance of Understanding Students' Construction in the Naming Fraction of the Three Representation. *Journal of Physics: Conference Series*, 1114(1).
- Samkof, A., Lai, Y., & Weber, K. (2012). On The Different Ways That Mathematicians Use Diagrams In Proof Construction. *Research in Mathematics Education*, 14(1), 49-67.
- Tall, D. (2008a). The transition to formal thinking in mathematics. *Mathematics Education Research Journal*, 20(2), 5-24.
- Tall, D. (2008b, May). *The Historical & Individual Development of Mathematical Thinking: Ideas that are Set-Before and Met-Before*. Paper presented at Colóquio de História e Tecnologia no Ensino Da Matemática, Rio de Janeiro, Brazil.
- Weber, K., & Alcock, L. (2004). Semantic and Syntactic Proof Productions. *Educational Studies in Mathematics*, 56(3), 209-234.